Fuzzy logic

Introduction 2

Fuzzy Sets & Fuzzy Rules

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Contents

- Characteristics of Fuzzy Sets
- Operations
- Properties
- Fuzzy Rules
- Examples

Characteristics of Fuzzy Sets

- The classical set theory developed in the late 19th century by Georg Cantor describes how crisp sets can interact. These interactions are called **operations**.
- Also fuzzy sets have well defined both **operations** and **properties**, as the basis for the fuzzy sets to deal with uncertainty on the one hand and to represent knowledge on the other.

Note: Membership Functions

- When a fuzzy set A is constructed over continuous universe of discourse X, it is described by its (continuous) membership function:

$$\mu_A(x), \quad x \in X$$

- When elements of a fuzzy set A belong to a finite universe of discourse:

$$X = \{x_1, x_2, .., x_n\},$$

usually a fuzzy set is denoted as:

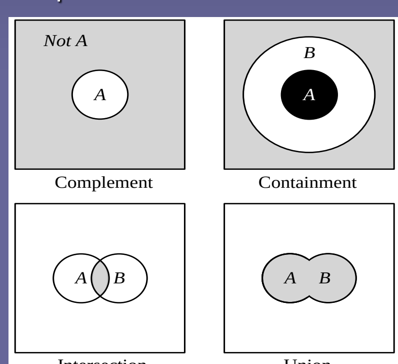
$$A = \mu_A(x_i)/x_i + \ldots\ldots\ldots\ldots + \mu_A(x_n)/x_n$$

where $\mu_A(x_i)/x_i$ (a singleton) is a pair: "grade of membership" /element.

3

Operations of Fuzzy Sets

- Elements either belong to crisp sets in full or don't belong at all.
- Operations of crisp sets:



- How do we make operations on fuzzy sets, which consist of elements belonging partially to them?

4

Complement

- Crisp Sets: Who does not belong to the set?
- Fuzzy Sets: How much do elements not belong to the set?

- The complement of a set is an opposite of this set. For example, if we have the set of tall men, its complement is the set of NOT tall men. When we remove the tall men set from the universe of discourse, we obtain the complement.

- If A is the fuzzy set, its complement $\neg A$ can be found as follows:
  $\mu_{\neg A}(x) = 1 - \mu_A(x)$.

5

Containment

- Crisp Sets: Which sets belong to which other sets?
- Fuzzy Sets: How much sets belong to other sets?

- Similar to a Chinese box, a set can contain other sets. The smaller set is called the **subset**. For example, the set of tall men contains all tall men; very tall men is a subset of tall men. However, the tall men set is just a subset of the set of men.
- In crisp sets, all elements of a subset entirely belong to a larger set.
- In fuzzy sets, however, each element can belong less to the subset than to the larger set. Elements of the fuzzy subset have smaller memberships in the subset than in the larger set.
- To be further discussed in Properties of fuzzy sets...

6

Intersection

- Crisp Sets: Which element belongs to both sets?
- Fuzzy Sets: How much of the element is in both sets?

- In classical set theory, an intersection between two sets contains the elements shared by these sets. For example, the intersection of the set of tall men and the set of fat men is the area where these sets overlap.
- In fuzzy sets, an element may partly belong to both sets with different memberships.

- A fuzzy intersection is the **lower membership** in both sets of each element. The fuzzy intersection of two fuzzy sets *A* and *B* on universe of discourse X:

  $\mu_{A \cap B}(x) = \min [\mu_A(x), \mu_B(x)] = \mu_A(x) \cap \mu_B(x)$,

  where $x \in X$.

7

Union

- Crisp Sets: Which element belongs to either set?
- Fuzzy Sets: How much of the element is in either set?

- The union of two crisp sets consists of every element that falls into either set. For example, the union of tall men and fat men contains all men who are tall OR fat.

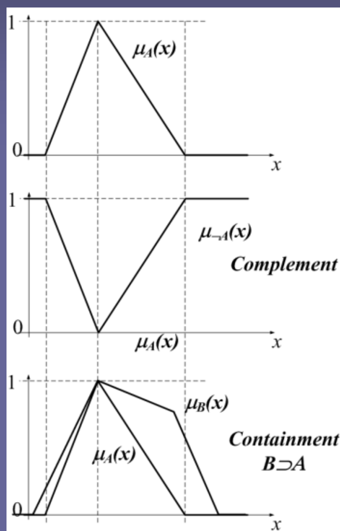
- In fuzzy sets, the union is the reverse of the intersection. That is, the union is the **largest membership** value of the element in either set. The fuzzy operation for forming the union of two fuzzy sets A and B on universe X can be given as:

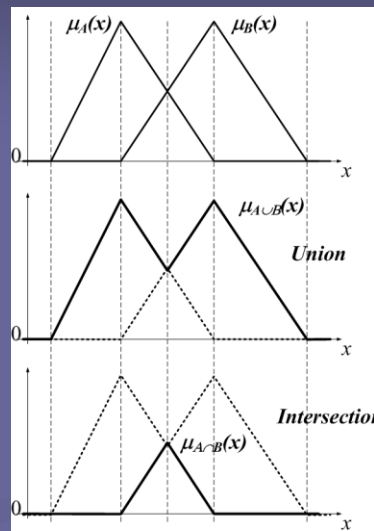
  $\mu_{A \cup B}(x) = \max [\mu_A(x), \mu_B(x)] = \mu_A(x) \cup \mu_B(x)$,

  where $x \in X$.

8

Operations of Fuzzy Sets





9

Properties of Fuzzy Sets

- Equality of two fuzzy sets
- Inclusion of one set into another fuzzy set
- Cardinality of a fuzzy set
- An empty fuzzy set
- α-cuts (alpha-cuts)

10

Equality

- Fuzzy set *A* is considered equal to a fuzzy set *B*, IF AND ONLY IF:

$$\mu_A(x) = \mu_B(x), \forall x \in X$$

- Example: $A = 0.3/1 + 0.5/2 + 1/3$, $B = 0.3/1 + 0.5/2 + 1/3$, therefore $A = B$.

Inclusion

- Inclusion of one fuzzy set into another fuzzy set. Fuzzy set $A \subseteq X$ is included in (is a subset of) another fuzzy set, $B \subseteq X$:

$$\mu_A(x) \leq \mu_B(x), \forall x \in X$$

- Example: Consider $X = \{1, 2, 3\}$ and sets *A* and *B*

  $A = 0.3/1 + 0.5/2 + 1/3$;
  $B = 0.5/1 + 0.55/2 + 1/3$

  then *A* is a subset of *B*, or $A \subseteq B$

Cardinality

- Cardinality of a crisp (non-fuzzy) set Z is the number of elements in Z. BUT the cardinality of a fuzzy set A, the so-called SIGMA COUNT, is expressed as a SUM of the values of the membership function of A, $\mu_A(x)$:

$$card_A = \mu_A(x_1) + \mu_A(x_2) + \ldots \mu_A(x_n) = \Sigma\mu_A(x_i), \quad \text{for } i=1..n$$

- Example: Consider $X = \{1, 2, 3\}$ and sets A and B

$A = 0.3/1 + 0.5/2 + 1/3;$
$B = 0.5/1 + 0.55/2 + 1/3$

$card_A = 1.8$
$card_B = 2.05$

13

Empty Fuzzy Set

- A fuzzy set A is empty, IF AND ONLY IF:

$$\mu_A(x) = 0, \forall x \in X$$

- Example: Consider $X = \{1, 2, 3\}$ and fuzzy set

$$A = 0/1 + 0/2 + 0/3,$$

then A is *empty*.

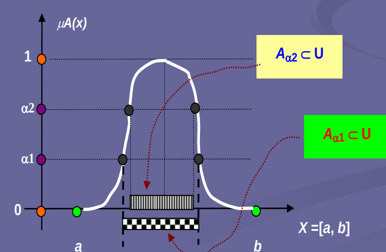
14

Alpha-Cut

- An α-cut or α-level set of a fuzzy set $A \subseteq X$ is an ORDINARY SET $A_\alpha \subseteq X$, such that:

$$A_\alpha=\{\mu_A(x)\geq\alpha,\ \forall x\in X\}.$$

- Example: Consider $X = \{1, 2, 3\}$ and set $A = 0.3/1 + 0.5/2 + 1/3$ then: $A_{0.5} = \{2, 3\}$, $A_{0.1} = \{1, 2, 3\}$, $A_1 = \{3\}$.
- Example: Consider continuous universe of discourse $X = [a, b]$ and fuzzy set A with the membership function $\mu_A(x)$. α-cuts for some α_1 *and* α_2 are:



Fuzzy Set Normality

- A fuzzy set A over universe X is called **normal** if there exists at least one element $x\in X$ such that $\mu_A(x) = 1$.
- A fuzzy set that is not normal is called **subnormal**.
- All crisp sets, except for the null set, are normal.
- In fuzzy set theory, the concept of nullness essentially generalises to subnormality.
- The **height** of a fuzzy set A is the largest membership grade of an element in A

$$height(A) = \max_x(\mu_A(x)).$$

- Fuzzy set is called **normal** if and only if:

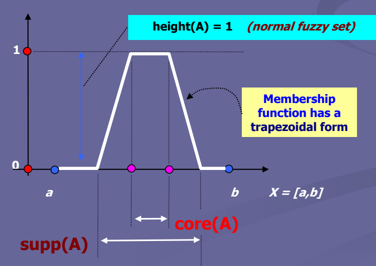
$$height(A) = 1.$$

Fuzzy Sets Core and Support

- Assume A is a fuzzy set over universe of discourse X.
- The **support** of A is the **crisp subset of X** consisting of all elements with membership grade:

  $supp(A) = \{x \mid \mu_A(x) > 0 \text{ and } x \in X\}$
- The **core** of A is the **crisp subset of X** consisting of all elements with membership grade:

  $core(A) = \{x \mid \mu_A(x) = 1 \text{ and } x \in X\}$
- Example:



Fuzzy Set Math Operations

- $kA = \{k\mu_A(x), \forall x \in X\}$

  Let k =0.5, and

  $A = 0.5/a + 0.3/b + 0.2/c + 1/d$

  then

  $kA = 0.25/a + 0.15/b + 0.1/c + 0.5/d$
- $A^m = \{\mu_A(x)^m, \forall x \in X\}$

  Let m =2, and

  $A = 0.5/a + 0.3/b + 0.2/c + 1/d$

  then

  $A^m = 0.25/a + 0.09/b + 0.04/c + 1/d$
- …

Fuzzy Sets Examples

- Consider two fuzzy set over the universe
  $X = \{a, b, c, d, e\}$,
  referred to as A and B, respectively:
  $A = 1/a + 0.3/b + 0.2/c + 0.8/d + 0/e$
  and
  $B = 0.6/a + 0.9/b + 0.1/c + 0.3/d + 0.2/e$

- Support:
  $supp(A) = \{a, b, c, d\}$
  $supp(B) = \{a, b, c, d, e\}$
- Core:
  $core(A) = \{a\}$
  $core(B) = \{\}$
- Cardinality:
  $card(A) = 1+0.3+0.2+0.8+0 = 2.3$
  $card(B) = 0.6+0.9+0.1+0.3+0.2 = 2.1$
- Complement:
  $A = 1/a + 0.3/b + 0.2/c + 0.8/d + 0/e$
  $\neg A = 0/a + 0.7/b + 0.8/c + 0.2/d + 1/e$
- Union:
  $A \cup B = 1/a + 0.9/b + 0.2/c + 0.8/d + 0.2/e$
- Intersection:
  $A \cap B = 0.6/a + 0.3/b + 0.1/c + 0.3/d + 0/e$

19

Fuzzy Sets Examples

- Again, two fuzzy sets over the universe $X = \{a, b, c, d, e\}$:
  $A = 1/a + 0.3/b + 0.2/c + 0.8/d + 0/e$ and
  $B = 0.6/a + 0.9/b + 0.1/c + 0.3/d + 0.2/e$

- kA:
  for k=0.5
  $kA = 0.5/a + 0.15/b + 0.1/c + 0.4/d + 0/e$

- A^m:
  for m=2
  $A^m = 1/a + 0.09/b + 0.04/c + 0.64/d + 0/e$

- α-cut:
  $A_{0.2} = \{a, b, c, d\}$
  $A_{0.3} = \{a, b, d\}$
  $A_{0.8} = \{a, d\}$
  $A_1 = \{a\}$

20

Fuzzy Rules

- In 1973, Lotfi Zadeh published his second most influential paper. This paper outlined a new approach to analysis of complex systems, in which Zadeh suggested capturing human knowledge in **fuzzy rules**.
- A fuzzy rule can be defined as a conditional statement in the form:

  IF *x* is *A* THEN *y* is *B*

  where *x* and *y* are linguistic variables; and *A* and *B* are linguistic values - fuzzy sets on the universe of discourses *X* and *Y*, respectively.
- For example:

  IF *project_duration* is *long* THEN *completion_risk* is *high*

  IF *speed* is *slow* THEN *stopping_distance* is *short*

21

Classical vs. Fuzzy Rules

- A classical IF-THEN rule uses binary logic, for example,

  Rule 1: IF *speed* >100 THEN *stopping_distance* = *long*

  Rule 2: IF *speed* < 40 THEN *stopping_distance* = *short*

- The variable *speed* can have any numerical value between 0 and 220 km/h, but the linguistic variable *stopping_distance* can take either value *long* or *short*.
- **In classical logic, rule is fired only if the antecedent is (strictly) true**, otherwise the consequent is not executed at all.
- In other words, classical rules are expressed in the black-and-white language of Boolean logic.

22

Classical vs. Fuzzy Rules

- We can also represent the stopping distance rules in a fuzzy form:

Rule 1: IF *speed* is *fast* THEN *stopping_distance* is *long*
Rule 2: IF *speed* is *slow* THEN *stopping_distance* is *short*

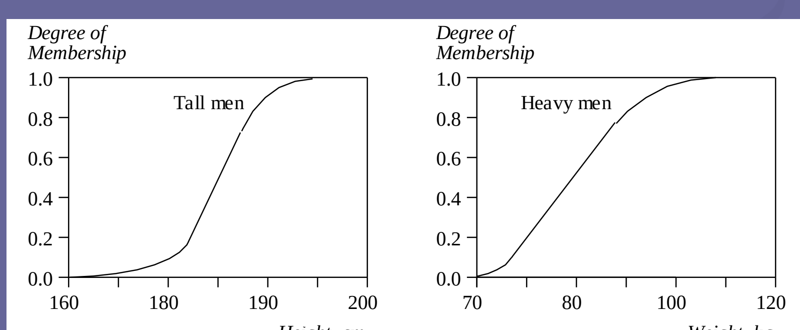
- In fuzzy rules, the linguistic variable *speed* also has the range (the universe of discourse) between 0 and 220 km/h, but this range includes fuzzy sets, such as *slow*, *medium* and *fast*.
- The universe of discourse of the linguistic variable *stopping_distance* can be between 0 and 300 m and may include such fuzzy sets as *short*, *medium* and *long*.
- Fuzzy rules relate fuzzy sets.
- **In a fuzzy system, all rules fire to some extent**, or in other words they fire partially.
- **If the antecedent is true to some degree** of membership, then **the consequent is also true to that same degree**.

23

Firing Fuzzy Rules

- These fuzzy sets provide the basis for a weight estimation model. The model is based on a relationship between a man's height and his weight:

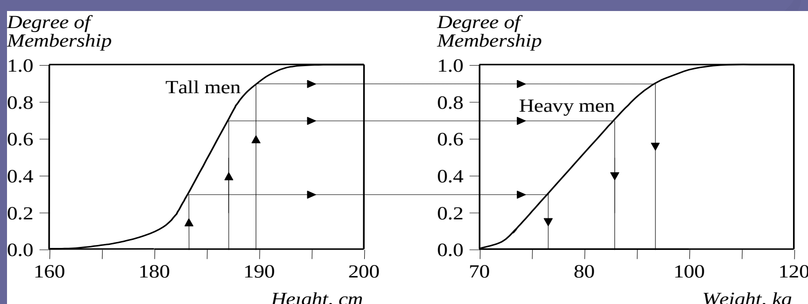
IF *height* is *tall* THEN *weight* is *heavy*



24

Firing Fuzzy Rules

- The value of the output or a truth membership grade of the rule consequent can be estimated directly from a corresponding truth membership grade in the antecedent. This form of fuzzy inference uses a method called **monotonic selection**.



Fuzzy Rules

- A fuzzy rule can have multiple antecedents, for example:

IF *project_duration* is *long* AND *project_staffing* is *large* AND *project_funding* is *inadequate*
THEN *risk* is *high*

IF *service* is *excellent* OR *food* is *delicious*
THEN *tip* is *generous*

- The consequent of a fuzzy rule can also include multiple parts, for instance:

IF *temperature* is *hot*
THEN *hot_water* is *reduced*;
cold_water is *increased*

Fuzzy Sets & Rules Example

- Air-conditioning involves the delivery of air which can be warmed or cooled and have its humidity raised or lowered.
- An air-conditioner is an apparatus for controlling, especially lowering, the temperature and humidity of an enclosed space. An air-conditioner typically has a fan which blows/cools/circulates fresh air and has cooler and the cooler is under thermostatic control. Generally, the amount of air being compressed is proportional to the ambient temperature.
- Consider Johnny's air-conditioner which has five control switches: COLD, COOL, PLEASANT, WARM and HOT. The corresponding speeds of the motor controlling the fan on the air-conditioner has the graduations: MINIMAL, SLOW, MEDIUM, FAST and BLAST.

27

Fuzzy Sets & Rules Example

- The rules governing the air-conditioner are as follows:

RULE 1:
IF TEMP is COLD THEN SPEED is MINIMAL

RULE 2:
IF TEMP is COOL THEN SPEED is SLOW

RULE 3:
IF TEMP is PLEASANT THEN SPEED is MEDIUM

RULE 4:
IF TEMP is WARM THEN SPEED is FAST

RULE 5:
IF TEMP is HOT THEN SPEED is BLAST

28

Fuzzy Sets & Rules Example

The **temperature** graduations are related to Johnny's perception of ambient temperatures.

where:

Y : *temp* value belongs to the set ($0<\mu_A(x)<1$)

Y* : *temp* value is the ideal member to the set ($\mu_A(x)=1$)

N : temp value is not a member of the set ($\mu_A(x)=0$)

Temp (^{0}C)	COLD	COOL	PLEASANT	WARM	HOT
0	Y*	N	N	N	N
5	Y	Y	N	N	N
10	N	Y	N	N	N
12.5	N	Y*	N	N	N
15	N	Y	N	N	N
17.5	N	N	Y*	N	N
20	N	N	N	Y	N
22.5	N	N	N	Y*	N
25	N	N	N	Y	N
27.5	N	N	N	N	Y
30	N	N	N	N	Y*

29

Fuzzy Sets & Rules Example

Johnny's perception of the **speed** of the motor is as follows:

where:

Y : *temp* value belongs to the set ($0<\mu_A(x)<1$)

Y* : *temp* value is the ideal member to the set ($\mu_A(x)=1$)

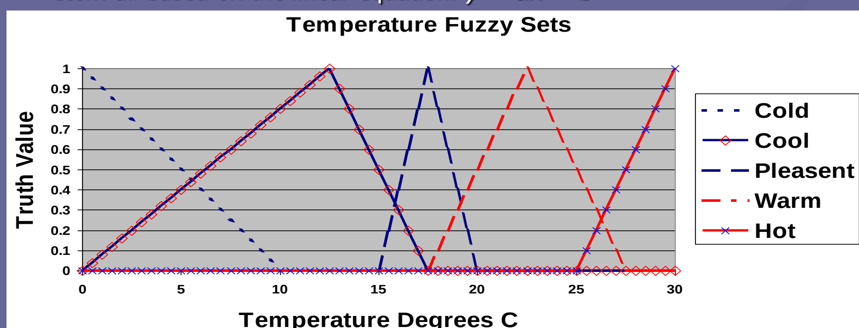
N : temp value is not a member of the set ($\mu_A(x)=0$)

Rev/sec (RPM)	MINIMAL	SLOW	MEDIUM	FAST	BLAST
0	Y*	N	N	N	N
10	Y	N	N	N	N
20	Y	Y	N	N	N
30	N	Y*	N	N	N
40	N	Y	N	N	N
50	N	N	Y*	N	N
60	N	N	N	Y	N
70	N	N	N	Y*	N
80	N	N	N	Y	Y
90	N	N	N	N	Y
100	N	N	N	N	Y*

30

Fuzzy Sets & Rules Example

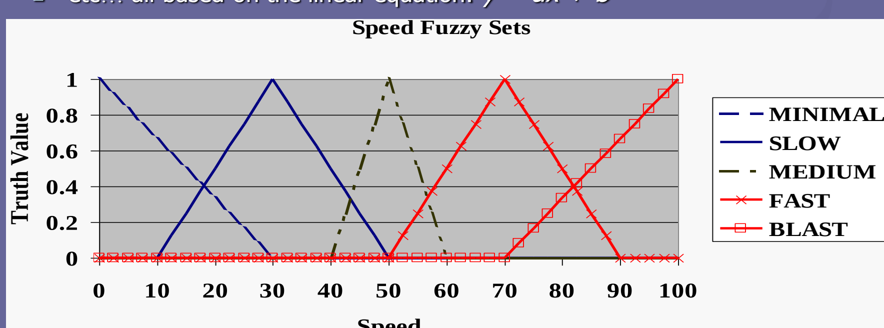
- The analytically expressed membership for the reference fuzzy subsets for the **temperature** are:
  - COLD: for $0 \le t \le 10$ $\mu_{COLD}(t) = -t / 10 + 1$
  - COOL: for $0 \le t \le 12.5$ $\mu_{COOL}(t) = t / 12.5$
    for $12.5 \le t \le 17.5$ $\mu_{COOL}(t) = -t / 5 + 3.5$
- etc… all based on the linear equation: $y = ax + b$



31

Fuzzy Sets & Rules Example

- The analytically expressed membership for the reference fuzzy subsets for the **speed** are:
- MINIMAL: for $0 \le v \le 30$ $\mu_{MINIMAL}(v) = -v / 30 + 1$
- SLOW: for $10 \le v \le 30$ $\mu_{SLOW}(v) = v / 20 - 0.5$
  for $30 \le v \le 50$ $\mu_{SLOW}(v) = -v / 20 + 2.5$
- etc… all based on the linear equation: $y = ax + b$



32

Exercises

For

$A = 0.2/a + 0.4/b + 1/c + 0.8/d + 0/e$

$B = 0/a + 0.9/b + 0.3/c + 0.2/d + 0.1/e$

calculate the following:

- Support, Core, Cardinality, and Complement for *A* and *B* independently,
- Union and Intersection of *A* and *B*,
- the new set $C = A^2$
- the new set $D = 0.5 \times B$
- the new set *E*, which is the alpha cut at $A_{0.5}$

33

Solutions

$A = 0.2/a + 0.4/b + 1/c + 0.8/d + 0/e$

$B = 0/a + 0.9/b + 0.3/c + 0.2/d + 0.1/e$

Support

Supp(*A*) = {*a*, *b*, *c*, *d*}

Supp(*B*) = {*b*, *c*, *d*, *e*}

Core

Core(*A*) = {*c*}

Core(*B*) = {}

Cardinality

Card(*A*) = 0.2 + 0.4 + 1 + 0.8 + 0 = 2.4

Card(*B*) = 0 + 0.9 + 0.3 + 0.2 + 0.1 = 1.5

Complement

Comp(*A*) = {0.8/*a*, 0.6/*b*, 0/*c*, 0.2/*d*, 1/*e*}

Comp(*B*) = {1/*a*, 0.1/*b*, 0.7/*c*, 0.8/*d*, 0.9/*e*}

Union

$A \cup B = 0.2/a + 0.9/b + 1/c + 0.8/d + 0.1/e$

Intersection

$A \cap B = 0/a + 0.4/b + 0.3/c + 0.2/d + 0/e$

Math Operations

$C = A^2$

$C = 0.04/a + 0.16/b + 1/c + 0.64/d + 0/e$

$D = 0.5 \times B$

$D = 0/a + 0.45/b + 0.15/c + 0.1/d + 0.05/e$

α-cut

$E = A_{0.5} = \{c, d\}$

34